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The appearance of a geometric-type instability in dynamic systems with adiabatically varying parameters

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Abstract. Dynamic systems which are described by homogeneous ordinary differential equations with slowly variable coefficients are considered. The eigenvalues of these equations correspond to stable states of the system. The exponential growth that has been discovered within the WKB approximation corresponds to a new type of instability. It results in a physically reversible, continuous energy transfer to the system due to adiabatic variation of two or more parameters. A general theory is presented and simple mathematical and physical examples are given. A particular case of the effect where the amplitude of the solutions does not change is the increase of the so-called geometric (Berry–Hannay) phase.

1. Introduction

Soon after the independent (and nearly simultaneous) appearance of the papers by Berry and Hannay [1–3] interest toward the so-called geometric or topological phases strongly increased in physics. This interest has proven to be valid as the geometric phases helped to explain and predict a series of various effects. The experimental and theoretical works connected with the geometric phase belong to quite different fields of physics, such as the classical oscillator, chemical reactions, solid state physics and ray optics, as well as quantum mechanics and superfields (see the review [4] and references therein). At first, the geometric phases were considered in a variety of physical systems with slowly varying parameters. Such systems often possess adiabatic invariants relating changes in the solution amplitude and the system energy with variation of the parameters. However, the existence of an adiabatic invariant does not pose restrictions on the phase variation of the solution. Therefore, a geometric change of phase may arise. It means that the change of the geometric phase component does not depend on the rate of parameter variation but is determined solely by the trajectory of the representative point in the parameter space. The geometric phases considered by Berry and Hannay are anholonomic or nonintegrable. This means that the change of the geometrical phase component is not equal to zero when the representative point moves along a closed contour. In particular, if the representative point moves continuously along a closed contour, the geometric phase may grow infinitely.

This paper is devoted to an analogous effect in the amplitude of the solution. The effect may display itself in systems with slowly varying parameters which have no adiabatic invariant (the existence of one would prohibit the unlimited growth of the system energy with finitely varying parameters). Using the WKB approximation (which is equal to the adiabatic) solutions

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of an arbitrary linear ordinary differential equation with adiabatic variation of the coefficients have been investigated. The geometric part of the solution phase has been isolated in the general case. In turn, this part has a component which is anholonomic. It is shown that the anholonomic geometric phase, generally speaking, is complex, even in equations with real coefficients. This means that anholonomic geometric terms may not only appear in the phase of the solution but in the amplitude as well. Hence, when the representative point of the system moves along a closed contour in the parameter space, the solution will either grow exponentially or decrease (which depends on the direction of motion along the contour). For this reason we will call the effect geometric instability, by analogy with the geometric phase.

Let us note the specific properties of this instability that do not allow us to include it in any of the well known classes. First, the eigenvalues of the equation may have no positive real part at every fixed time moment. This means that the eigenvalues correspond to stable states of the system. Second, the instability cannot be considered a parametric resonance. Indeed, parametric resonances appear when coefficients of an equation change periodically and at certain frequencies. Moreover, in a parametric resonance one of the solutions grows exponentially regardless of the phase coefficient variation. In this sense the parametric resonance instability is physically irreversible. Meanwhile, in the presence of a geometric instability the amplitude growth is determined only by the representative point in the parameter space. It does not depend upon the rate of coefficient variations. Thus, the geometric instability should be observed with aperiodic variation of the equation coefficients. If the coefficients vary periodically, the variation frequency may be arbitrarily low. Finally, the geometric instability is physically reversible. If the equation solutions showed exponential growth with a certain type of a representative point motion along a contour, then they should decrease if the point moves in the opposite direction: the instability changes into damping and the system will return to its initial state.

Along with a general analysis, specific mathematical and physical examples are considered. It is shown that geometric instability may manifest itself even in the simplest of differential equations. The theoretical derivations are corroborated by numerical analysis of the equations which is in agreement with the theory. The WKB approach used in this paper is an alternative, natural for linear adiabatic systems, to the averaging method or that of reduction to normal form [5, 6] which are generally applied to analyse the geometric phase. The equivalence of the WKB method to the formalism employed by other workers is shown in the appendix by considering the known example of the generalized oscillator.

2. General theory

Let us consider a dynamic system which is described by a homogeneous linear ordinary differential equation of order n, with slowly varying coefficients,

$$y^{(n)}(x) + \sum_{k=1}^{n} a_k(\varepsilon x) y^{(n-k)}(x) = 0. \tag{1}$$

Here $\varepsilon \ll 1$ is the adiabatic variation parameter. We will assume that equation (1) has no singular points in the domain of variation of the coefficients. Then its solutions may be obtained within the first approximation in ε, by the well known WKB method (see, for instance [7, 8]).

Let the state of the initial system be determined by coordinates of the representative point in the n-dimensional phase space and the set of adiabatically variable parameters $q = (q_1(\varepsilon x), \dots, q_s(\varepsilon x))$. Then the coefficients of equation (1), generally speaking, will contain small derivatives $|q_\alpha^{(j)}(\varepsilon x)| \sim \varepsilon^j |q_\alpha(\varepsilon x)|$ (here and further below the differentiation is with respect to x). The WKB approximation suggests retaining only the zeroth and first-order terms in ε. Then equation (1) can be written in the form

$$y^{(n)}(x) + \sum_{k=1}^{n} a_{k0}(\varepsilon x) y^{(n-k)}(x) + \sum_{k=1}^{n} a_{k1}(\varepsilon x) y^{(n-k)}(x) = 0. \tag{2}$$

Where the coefficients a_{k0} depend only on the set of parameters q and are of order one, while the coefficients a_{k1} are proportional to first derivatives of the parameters and have an order of ε,

$$\begin{aligned} a_{k0}(\varepsilon x) &= \tilde{a}_{k0}(q(\varepsilon x)) \\ a_{k1}(\varepsilon x) &= \sum_{\alpha=1}^{s} \tilde{b}_{k\alpha}(q(\varepsilon x)) q'_{\alpha}(\varepsilon x). \end{aligned} \tag{3}$$

By substituting $y(x) = \exp(px)$ into (2) we arrive at the characteristic equation

$$\begin{aligned} l(\varepsilon x, p) &= l_0(\varepsilon x, p) + l_1(\varepsilon x, p) = 0 \\ l_0(\varepsilon x, p) &\equiv p^n + \sum_{k=1}^{n} a_{k0}(\varepsilon x) p^{n-k} \\ l_1(\varepsilon x, p) &\equiv \sum_{k=1}^{n} a_{k1}(\varepsilon x) p^{n-k}. \end{aligned} \tag{4}$$

Here the left-hand side of the general characteristic equation has been separated into two parts which correspond to different orders of ε. The set of independent WKB solutions of equations (1) and (2) can be written in the form of [7]:

$$y_j(x) = \exp\left\{ \int^x p_j(\varepsilon\xi)\, \mathrm{d}\xi - \frac{1}{2} \int^x \frac{l''_{pp}(\varepsilon\xi, p_j(\varepsilon\xi))}{l'_p(\varepsilon\xi, p_j(\varepsilon\xi))} p'_j(\varepsilon\xi)\, \mathrm{d}\xi \right\} \qquad j = 1, \ldots, n. \tag{5}$$

Here $p_j(\varepsilon x)$ are the roots of the characteristic equation (4). Let us separate small terms of different order in (5). The second term in the exponent of (5) is proportional to derivatives of the roots of the characteristic equation, and hence is of order ε. The first term, as is easy to see from (4), includes terms of both order zero and order one in ε. First, we will look for approximate solutions $p_j(\varepsilon x)$. Within zeroth order in ε, the roots of (4) are determined from the truncated equation $l_0(x, p) = 0$. They can be written in the form (see (3) and (4))

$$p_{j0}(\varepsilon x) = \tilde{p}_{j0}(q(\varepsilon x)). \tag{6}$$

By applying the perturbation method it is easy to obtain solutions of equation (4) up to first-order terms in ε,

$$\begin{aligned} p_j(\varepsilon x) &= p_{j0}(\varepsilon x) + p_{j1}(\varepsilon x) \\ p_{j1}(\varepsilon x) &\equiv -\frac{\sum_{k=1}^{n} a_{k1}(\varepsilon x) p_{j0}^{n-k}(\varepsilon x)}{n p_{j0}^{n-1}(\varepsilon x) + \sum_{k=1}^{n} (n-k) a_{k0}(\varepsilon x) p_{j0}^{n-k-1}(\varepsilon x)}. \end{aligned} \tag{7}$$

Using (3) and (6) it is possible to represent the solutions (7) as functions of the parameters $q(\varepsilon x)$ and their small derivatives,

$$\begin{aligned} p_j(\varepsilon x) &= \tilde{p}_j(q(\varepsilon x), q'(\varepsilon x)) = \tilde{p}_{j0}(q(\varepsilon x)) + \tilde{p}_{j1}(q(\varepsilon x), q'(\varepsilon x)) \\ \tilde{p}_{j1}(q, q') &\equiv -\frac{\sum_{\alpha=1}^{s} \sum_{k=1}^{n} \tilde{p}_{j0}^{n-k}(q) \tilde{b}_{k\alpha}(q) q'_{\alpha}}{n \tilde{p}_{j0}^{n-1}(q) + \sum_{k=1}^{n} (n-k) \tilde{a}_{k0}(q) \tilde{p}_{j0}^{n-k-1}(q)}. \end{aligned} \tag{8}$$

The truncated characteristic equation $l_0(x, p) = 0$ can be represented in the form of (see (4) and (6))

$$l_0(x, p) \equiv \prod_{l=1}^{n} (p - \tilde{p}_{l0}(q(\varepsilon x))) = 0. \tag{9}$$

Now, the WKB solutions of (5) can be written in a more convenient form. By substituting the solutions (6) and (8) and equations (4) and (9) into (5) and linearizing the exponent of (5) with respect to ε, we obtain

$$y_j(x) = \exp\left\{ \int^x p_{j0}(\varepsilon\xi)\,\mathrm{d}\xi + \Phi_j(\varepsilon x) \right\} \qquad j = 1, \ldots, n \tag{10}$$

with

$$\Phi_j(x) = \int^{q(x)} \sum_{\alpha=1}^{s} G_{j\alpha}(q)\,\mathrm{d}q_\alpha$$

and

$$G_{j\alpha}(q) \equiv \frac{\sum_{k=1}^{n}\left(\sum_{l\neq j} \frac{\tilde{p}_{0j}^{n-k}(q)\,\frac{\partial \tilde{a}_{k0}(q)}{\partial q_\alpha}}{\tilde{p}_{0j}(q)-\tilde{p}_{0l}(q)} - \tilde{p}_{0j}^{n-k}(q)\tilde{b}_{k\alpha}(q) \right)}{n\tilde{p}_{0j}^{n-1}(q) + \sum_{k=1}^{n}(n-k)\tilde{a}_{k0}(q)\tilde{p}_{0j}^{n-k-1}(q)}. \tag{11}$$

As can be seen from (10) and (11), the exponent of the WKB solution contains, along with the ordinary dynamic phase $\int^x p_{j0}(\varepsilon\xi)\,\mathrm{d}\xi$, a complex geometric phase Φ_j. The value of the dynamic phase depends explicitly on x, while the value of Φ_j does not show an explicit dependence on x, being determined by an integral of $G_j(q) = (G_{j1}(q), \ldots, G_{js}(q))$ along the representative point trajectory in the s-dimensional space of parameters. Thus, the geometric phase does not depend on the adiabatic approximation on the rate of parameter variation but solely on the representative point trajectory in the q-space. In the simpler case of a single variable parameter ($s = 1$) the field $G_j(q)$ is an analytic function having a primitive (we consider parameter variations far from singularities). Then the phase Φ_j is holonomic, i.e. it depends only on the current value of the parameter and does not depend on the contour of integration. In this case the second term in the exponent of the WKB solution (10) can be used to construct the adiabatic invariant of the system.

But, in the general case of two or more independently varying parameters ($s \geqslant 2$) the primitive may be absent, or it is impossible to introduce a potential for $G_j(q)$, since the field may also have a curl component. The potential component of $G_j(q)$ possesses a primitive and determines the holonomic part of the geometric phase Φ_j. The curl component of $G_j(q)$ determines the anholonomic part of Φ_j which depends essentially on the integration contour in the q-space. If the system trajectory in the q-space is a closed contour, then parameter values at the starting point and endpoint are the same but the anholonomic phase gained along the path is not equal to zero and the system does not return to the initial state.

Similar effects were first considered by Berry and Hannay [1–3], however they and later writers only investigated the imaginary gain of the anholonomic phase Φ_j, and therefore the solution amplitude did not change as a result of the representative point motion along a closed contour.

It follows from the general equations (10) and (11) that the phase gained along the path is complex, and hence the anholonomic component of Φ_j may have a real part. This case is of special interest representing the possibility of reversible change of the system energy (here and below we assume that the system energy is a monotonous function of the solution amplitude).

Consider the simplest case where the representative point moves along a closed contour μ in the q-space. Then the amplitude of the jth WKB solution increases after one revolution by the factor $\exp\{\mathrm{Re} \oint_\mu G_{j\alpha}(q)\,\mathrm{d}q_\alpha\}$. Continuous motion along μ results in an exponential growth or exponential decay of the solution, depending on the sign of the real part of the phase gained over one period. Physically, the process is completely reversible: if the representative point moves in the opposite direction around the same μ, then the sign of the phase gained in the exponent will change and the system will evolve in the opposite direction.

It should be especially emphasized that motion along a closed contour in the q-space does not imply a periodic variation of the coefficients. They may depend on x in an arbitrary way (though at a sufficiently slow rate). The values they assume ought to obey the contour equation. In the case of a periodic variation of $q(\varepsilon x)$ the phase gained over one period does not depend on the frequency. Thus, the geometric instability effect cannot be considered as a parametric resonance.

The growth rate of the jth solution, γ_j, with a periodically varying $q(\varepsilon x)$ (period T) is

$$\gamma_j = \frac{1}{T}\,\mathrm{Re}\oint_{\mu} G_{j\alpha}(q)\,\mathrm{d}q_\alpha. \tag{12}$$

It is easily seen that the order of magnitude of γ_j is $\varepsilon \sim 1/T$. In the case of small parameter variation amplitudes, the rate γ_j, is, according to the Stokes theorem, proportional to the area of a surface stretched on the contour μ, that is to the squared amplitude of parameter variation. Since the growth rate γ_j of the instability discovered is always small compared with the eigenvalue magnitudes p_j of the initial equation (1), it seems natural to look for manifestations of the effect in such equations which do not possess eigenvalues with a positive real part (i.e. all the eigenvalues correspond to bounded solutions). The characteristic values may be purely imaginary or have small (much smaller than ε) real parts. It is the solutions that correspond to such eigenvalues where the geometric instability should be essential.

It should be noted that the contour μ should not necessarily be closed. An exponential growth may exist for fairly complex open trajectories. The sign of the phase increment is determined by the direction of circular motion with respect to the curl component of the field $G_j(q)$.

In conclusion, let us examine the natural question as to which systems may support the geometric instability. Let us assume for a moment that the effect is present in the system if the motion of a representative point around a closed contour in the q-space results in a finite gain of the geometric phase,

$$\Phi_{j0} = \oint_{\mu} G_{j\alpha}(q)\,\mathrm{d}q_\alpha \neq 0 \tag{13}$$

although Φ_{j0} may be purely imaginary and have no effect on the solution amplitude. Mathematically, the condition (13) means that the curl of the s-dimensional field $G_j(q)$ is not equal to zero, i.e.

$$(\mathrm{curl}\,G_j(q))_{\alpha\beta} = \frac{\partial G_{j\alpha}(q)}{\partial q_\beta} - \frac{\partial G_{j\beta}(q)}{\partial q_\alpha} \neq 0. \tag{14}$$

The effect under consideration will exist if the initial equation is not invariant with respect to change in the sign of the independent variable x (this condition is necessary but not sufficient). Really, if the sign of x changes, the representative point of the system starts moving along the contour in the opposite direction, and hence the phase gain (13) changes its sign. Therefore, the system satisfying (13) is not invariant with respect to changes of sign of the independent variable. With regard to the eigenvalues $p_j(\varepsilon x)$ this condition means that the points $p_j(\varepsilon x)$ in the complex plane have no fixed centre of inversional symmetry. Actually, if such a symmetry centre existed, then the substitution $y(x) = y_1(x)\exp(p_c x)$ (where p_c is a complex value corresponding to the symmetry centre location) would allow displacement of the entire set of eigenvalues in the complex plane in such a way as to make the origin of coordinates the symmetry centre. Apparently, in that case the initial equation would be invariant with relative changes of the sign of x.

3. Simple mathematical examples

In this section some examples of the simple differential equations are given, where the above considered geometric instability may prove observable. For the sake of simplicity, we will only consider the equations whose coefficients depend on the parameters but not their derivatives. All the derivations for concrete equations which were given by the general scheme of section 2 are omitted here. Details of the WKB formalism can be found in the appendix, with calculations of the geometric phase for the known example of a generalized oscillator.

3.1. Geometric instability in equations with real coefficients

As was shown in section 2, it is necessary for the instability effect that the eigenvalues should not have a fixed point of inversion symmetry in the complex plane. The eigenvalues of an equation with real coefficients always lie symmetrically with respect to the real axis. Therefore, it is necessary that the eigenvalues should not possess a fixed symmetry parallel to the imaginary axis. For the case of a real second-order equation this means that its eigenvalues should have a variable real part, which corresponds to a variable growth or decay rate in the WKB solutions (we take for granted that the real part of the eigenvalues does not change its sign). Obviously, the magnitude of the effect under consideration (variation of the phase Φ_{j0}) will be proportional to the amplitude of variation of this growth/decay rate. When the amplitude reduces to zero the eigenvalues acquire a fixed axis of the symmetry, such that the effect is certainly absent. In addition, the phase gain is proportional to the small parameter ε, which implies that the growth rate of the geometric instability for a real second-order equation, which is associated with Φ_{j0}, will always be much smaller than the 'principal' growth/decay rate connected with the real part of the eigenvalue.

Strictly speaking, if the real part of an eigenvalue is proportional to the derivative of a parameter and changes its sign with oscillations of the parameter, then the effect may be possible in second-order equations. We will not give the corresponding examples here, nor consider equations containing parameter derivatives.

Thus, it is possible to state that the lowest order of a real equation in which the geometric instability may exist is equal to three (a first-order equation depends on a single parameter, while $\Phi_{j0} \neq 0$ is possible if there are at least two parameters (see section 2).

Consider the equation

$$y''' + \chi(\varepsilon x)y'' + \omega^2(\varepsilon x)y' + \chi(\varepsilon x)\omega^2(\varepsilon x)y = 0. \qquad (15)$$

The corresponding characteristic equation can be written as

$$l(\varepsilon x, p) = (p - \mathrm{i}\omega(\varepsilon x))(p + \mathrm{i}\omega(\varepsilon x))(p + \chi(\varepsilon x)) = 0. \qquad (16)$$

It is not difficult to see that two roots of equation (16) correspond to oscillating solutions, while the third root to a damping one. Thus, at first sight equation (15) does not have growing solutions (we exclude the case of parametric resonances). Using the general formalism of section 2 and the final formula (11), we will derive expressions for the phases $\Phi_{1,2}$ corresponding to oscillating solutions of (15), namely:

$$\Phi_{1,2} = -\int \frac{\pm 3\mathrm{i}\omega + \chi}{\pm 2\mathrm{i}\omega(\pm\mathrm{i}\omega + \chi)}\,\mathrm{d}(\pm\mathrm{i}\omega).$$

The real part of the anholonomic part of the phase is

$$\mathrm{Re}\,\Phi_{1,2} = -\int \frac{\omega}{\omega^2 + \chi^2}\,\mathrm{d}\omega \neq 0. \qquad (17)$$

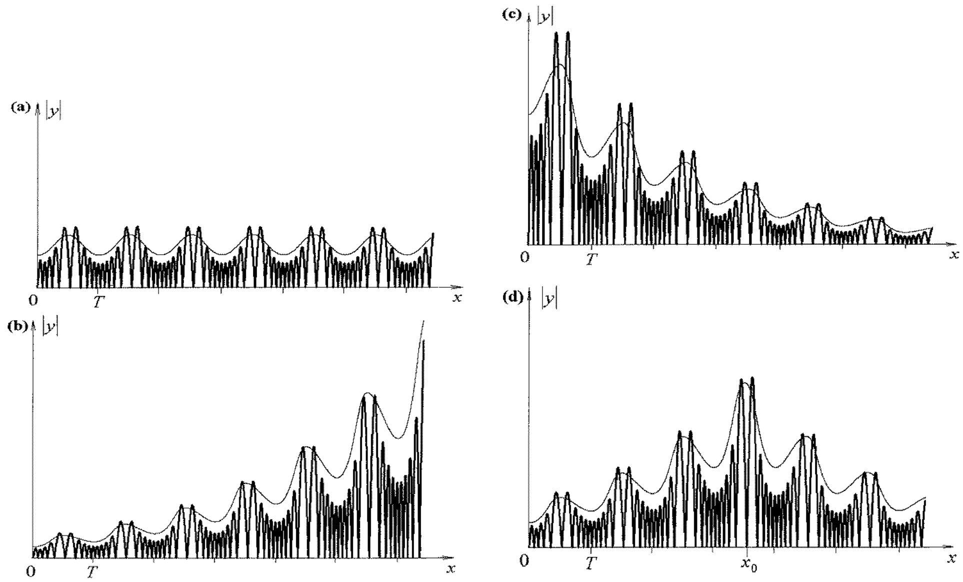

Figure 1. Amplitudes $|y(x)|$ of the solution to equation (15) for different variable parameters $\omega(\varepsilon x) = 1 + e_1 \cos(\varepsilon x)$, $\chi(\varepsilon x) = 1 + e_2 \sin(\varepsilon x)$ (bold curve). The thin curve represents the $\exp(\mathrm{Re}\,\Phi_{1,2}(x))$ dependence calculated after equation (17). As can be seen, the amplitude variations arising from changes of the parameters are determined by the phase $\mathrm{Re}\,\Phi$. The theoretical results are in full agreement with the computer simulation. The small-scale oscillations correspond to a sum of two complex WKB solutions of equal amplitudes and progressive phases $\pm\mathrm{i}\int^x \omega(\varepsilon\xi)\,\mathrm{d}\xi$. The period of large-scale oscillations coincides with that of parameter variations, $T = 2\pi\varepsilon^{-1}$. (*a*) $e_1 = 0.5$, $e_2 = 0$, $\varepsilon = 0.2$. The point (ω, χ) moves along a straight line and does not cover any area in the parameter space. Therefore, the phase Φ_{j0} gained over one period is equal to zero. (*b*) $e_1 = 0.5$, $e_2 = -0.5$, $\varepsilon = 0.2$. The point (ω, χ) moves around a circle in a clockwise direction. This produces an exponential growth of two independent WKB solutions. (*c*) $e_1 = 0.5$, $e_2 = 0.5$, $\varepsilon = 0.2$. The point (ω, χ) moves counterclockwise around the same circle. This brings forth an exponential decay of the two independent WKB solutions. (*d*) First $e_1 = 0.5$, $e_2 = -0.5$, $\varepsilon = 0.2$ the solution grows. At point $x = x_0$ the phase $\chi(\varepsilon x)$ is shifted by π, which corresponds to $e_1 = 0.5$, $e_2 = 0.5$, $\varepsilon = 0.2$. The solution attenuates to return to the initial condition at $x = 2x_0$. This example illustrates the reversibility of geometric instability.

Thus, the oscillating solutions which correspond to two purely imaginary roots of (16) begin to simultaneously grow or decrease with such variation of the parameters when the point (ω, χ) runs along a closed two-dimensional curve.

Figure 1 shows the results of solving equation (15) numerically with $\omega(\varepsilon x) = \omega_0 + e_1 \cos(\varepsilon x)$ and $\chi(\varepsilon x) = \chi_0 + e_2 \sin(\varepsilon x)$, which corresponds to the motion of the representative point (ω, χ) around a circle. Indeed, we are able to observe the simultaneous growth or damping of the two solutions at a rate corresponding to the calculation after equations (12) and (17). If the sign of e_2 in front of the sine was changed, then the representative point (ω, χ) would move along the contour in the opposite direction, and the evolution of the solutions will also proceed in the opposite direction. This means that the growth and damping of the solution may be controlled and reversed by changing the phase of the periodical dependence in one coefficient (figure 1(*d*)).

During the numerical solution of equation (15), a special check showed that the instability growth rate is strictly proportional to $\varepsilon \ll 1$, in accord with (12). In other words, the phase (17) gained after one turn of point (ω, χ) does not depend on the point velocity.

Note that introduction of a weak attenuation in solutions of the third-order equations (15) and (16) i.e. $p_1 = \mathrm{i}\omega_1 - \delta$ and $p_2 = -\mathrm{i}\omega - \delta$ with $\delta \ll \varepsilon$ does not change the general pattern. This result has also been verified numerically.

3.2. Geometric instability in a second-order equation with complex coefficients

The eigenvalues of an equation with complex coefficients lie asymmetrically relative to the real axis. Therefore, the geometric instability may appear in a complex equation of order as low as the second.

Consider the equation

$$y'' - \mathrm{i}(\omega_1(\varepsilon x) + \omega_2(\varepsilon x))y' - \omega_1(\varepsilon x)\omega_2(\varepsilon x)y = 0. \tag{18}$$

The corresponding characteristic equation may be represented as

$$l(\varepsilon x, p) = (p - \mathrm{i}\omega_1(\varepsilon x))(p - \mathrm{i}\omega_2(\varepsilon x)) = 0.$$

It can be seen that the roots of the characteristic equation correspond to oscillating solutions. The phases of the two solutions are

$$\Phi_{1,2} = -\int \frac{\mathrm{d}\omega_{1,2}}{\omega_{1,2} - \omega_{2,1}} \neq 0. \tag{19}$$

These are purely real, and the values gained as a result of one turn of the point (ω_1, ω_2) have opposite signs. This means that the amplitude of one solution will grow, while that of the other will attenuate. Should the point (ω_1, ω_2) move in the opposite direction, the damping and instability change place. In either case the amplitude of the general solution will grow exponentially.

Figures 2(*a*)–(*c*) show the results of numerical solution of equation (18) with $\omega_1(\varepsilon x) = \omega_{10} + e_1 \cos(\varepsilon x)$ and $\omega_2(\varepsilon x) = \omega_{20} + e_2 \sin(\varepsilon x)$. Figures 2(*d*)–(*f*) demonstrate the spectra of these solutions. It is seen that, depending on the direction of motion of point (ω_1, ω_2), the growing solution is either that of frequency ω_1, or of ω_2.

4. Simple physical examples

The simplest physical systems which are described by linear differential equations may often be represented as sets of interacting oscillators of a different kind. In this section we will consider examples of such systems.

4.1. Coupled oscillators with attenuation

As is known, a system of interacting oscillators can be described in terms of a Hamiltonian which is invariant to relative changes of sign of the independent variable (time). In order to obtain the desired effect of geometric instability it is necessary that the system should not be invariant with respect to time inversion (section 2). A noninvariance of the system will arise if we introduce damping. At least one oscillator ought to be only weakly damping, such that the geometric instability with a small growth rate ($\sim\varepsilon$) could be observed. It is also necessary that one of the oscillators should be strongly damping (rate of order 1). If the damping of all oscillators were weak, then the geometric instability growth rate would be of order $\chi\varepsilon \ll \chi$ (where χ is the small damping rate), and the effect would not be observable (see section 3.1).

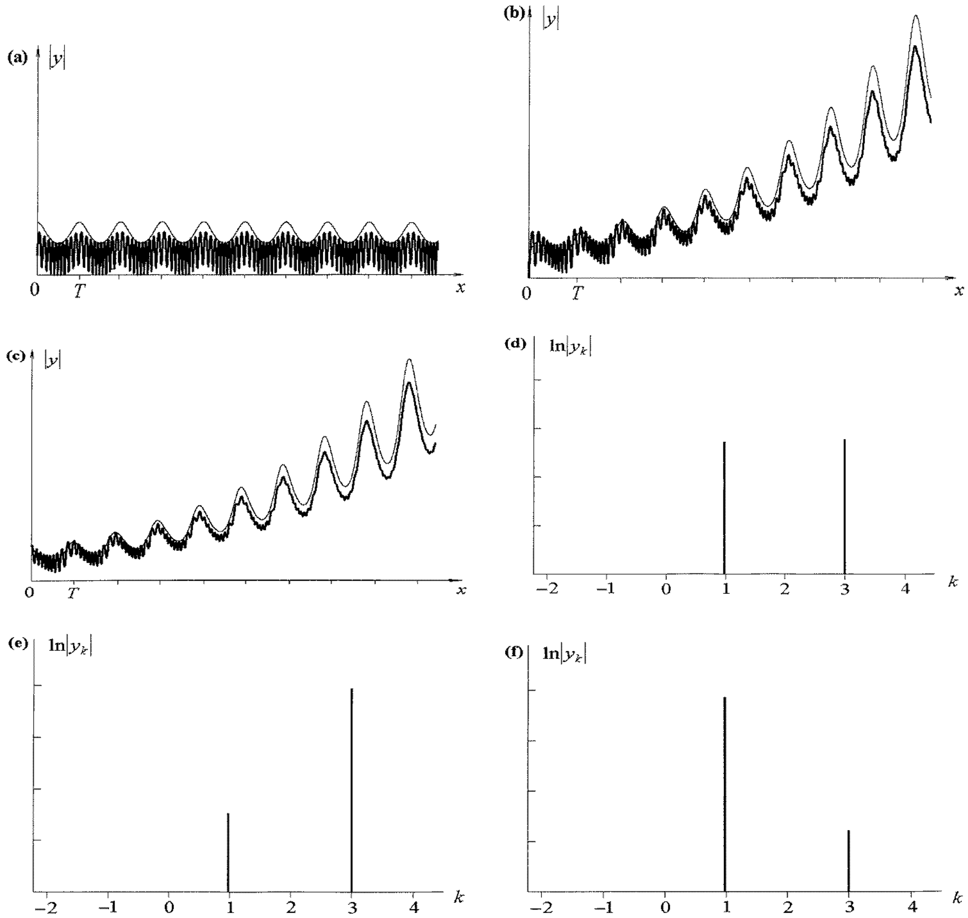

Figure 2. (*a*)–(*c*) Amplitudes $|y(x)|$ of the solution to equation (18) for different variable parameters $\omega_1(\varepsilon x) = 1 + e_1\cos(\varepsilon x)$, $\omega_2(\varepsilon x) = 3 + e_2\sin(\varepsilon x)$ (bold curve). The thin curve corresponds to the $\exp(\max\{\mathrm{Re}\,\Phi_{1,2}(x)\})$ dependence calculated after equation (19). The small-scale oscillations correspond to a sum of two complex WKB solutions of different amplitudes with progressive phases $\mathrm{i}\int^x \omega_{1,2}(\varepsilon\xi)\,\mathrm{d}\xi$. The period of large-scale oscillation coincides with that of parameter variation, $T = 2\pi\varepsilon^{-1}$. (*d*)–(*f*): The spectra $y_k(k) = \frac{1}{2\pi}\int y(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x$ of solutions to equation (18) for fixed coefficients $\omega_1 = 1$, $\omega_2 = 3$, after the coefficients of equation (18) have been altered at some x-interval in accord with (*a*)–(*c*) respectively. (*a*), (*d*) $e_1 = 0.5$, $e_2 = 0$, $\varepsilon = 0.2$. The point (ω_1, ω_2) moves along a straight line and does not cover any area in the parameter space. Accordingly, the phase Φ_{j0} gained over one period is equal to zero. The amplitudes of the two independent WKB solutions do not change. (*b*), (*e*) $e_1 = 0.5$, $e_2 = -0.5$, $\varepsilon = 0.2$. The point (ω_1, ω_2) moves around a circle in a clockwise direction which causes an exponential growth of the WKB solution subscripted 2 and exponential decay of the WKB solution 1. (*c*), (*f*) $e_1 = 0.5$, $e_2 = 0.5$, $\varepsilon = 0.2$. The point (ω_1, ω_2) moves counterclockwise around the same circle, causing an exponential growth of the solution subscripted 1 and exponential decay of solution 2.

Let us consider two coupled oscillators. One is characterized by strong damping due to friction ($\chi \sim 1$), while the other oscillates without friction but is subject to weak damping through the transfer of energy to the first oscillator. The initial equation set is

$$\begin{aligned} &y_1'' + \omega_1^2(\varepsilon x)y_1 = \nu_1(\varepsilon x)y_2 \\ &y_2'' + \chi(\varepsilon x)y_2' + \omega_2^2(\varepsilon x)y_2 = -\nu_2(\varepsilon x)y_1. \end{aligned} \tag{20}$$

The corresponding characteristic equation can be written, within zero-order approximation in ε, as

$$l_0 = p^4 + \chi p^3 + (\omega_1^2 + \omega_2^2)p^2 + \chi\omega_1^2 p + (\omega_1^2\omega_2^2 + \nu_1\nu_2) = 0.$$

It is easy to see that two roots of this equation always have a negative real part (damping rate of the solution) of order χ. The negative real part of the other two roots may be made as small as is desired together with the parameter $\nu_1\nu_2$, which is responsible for the oscillator coupling. However, the geometric instability growth rate will be small in this case, even compared with the small damping proportional to $\nu_1\nu_2$. This can be demonstrated rigorously, using equation (20) and the general formula (11), but we will not quote the lengthy derivations here, confining ourselves to a qualitative explanation of the result.

Really, the term $\nu_1\nu_2$ is responsible for oscillator coupling. With $\nu_1\nu_2 = 0$, the oscillators are independent. For each independent oscillator described by a real second-order equation the effect is absent (see section 3.1, as well as the appendix and [9], where the equivalence of a generalized oscillator and a damping oscillator is proved). Therefore, the growth rate of the geometric instability is proportional to the small parameter $\nu_1\nu_2$ and to the other small parameter, ε. Thus, the growth rate is $\sim \varepsilon\nu_1\nu_2 \ll \nu_1\nu_2$, which is what needed to be proved.

This example can be extended to the case of an arbitrary number of damping oscillators. Consequently, the effect is always small compared with damping and the solution cannot grow.

4.2. Moving coupled oscillators. The plasma–beam system

The asymmetry of the equation with regard to the sign change of the independent variable (time) for coupled oscillators can be introduced by means of the relative motion of the oscillators. Let one of the oscillators be at rest. The frequency of the other will be shifted owing to the Doppler effect. Then the two eigenvalues, $\pm\mathrm{i}\omega$, of the moving oscillator will receive the same imaginary addendum. In section 4.1 we achieved an asymmetry in the set of eigenvalues by means of displacing them along the real axis. Now the effect under consideration may arise due to eigenvalue displacement along the imaginary axis. Since it is difficult to imagine a continuous coupling of moving discrete oscillators, we shall consider distributed oscillating systems with relative motion. An example is given by the well known plasma–beam interaction.

In the simplest one-dimensional case the plasma represents an undulating system with the dispersion law $\omega = \pm\omega_0(k)$, where k is the wavenumber. The beam eigenwaves are characterized by a similar dispersion law shifted due to the Doppler effect: $\omega = kv \pm \omega_b$, where v is the velocity of beam particles, and ω_b the plasma frequency (Langmuir frequency) of the beam. Let $\omega_b \ll \omega_0(k)$. Then the frequencies of one of the plasma waves and the two beam waves become close for such $k = k_0$ that $\omega_0(k_0) = k_0 v$. The three waves start interacting, the beam waves being at resonance with the plasma wave. As a result one of the waves grows exponentially along the beam propagation direction (we consider a boundary problem with fixed frequencies of the waves). This effect is well known as the plasma–beam instability (PBI) (see, for instance [10]).

The differentiation operators reduce to multiplication by $\mathrm{i}\omega$, owing to uniformity of the system with respect to time. Then the initial wave equations in partial derivatives become ordinary equations with oscillating solutions, where the independent variable is the longitudinal coordinate. The frequency of the monochromatic waves is a fixed parameter of the problem. The basic equations which describe the linear stage of three wave interaction under the PBI conditions may be written in the form of [11],

$$\begin{aligned} &z' - \mathrm{i}\chi(\varepsilon x)z = -\mathrm{i}\rho(\varepsilon x)y \\ &y'' = \tfrac{1}{2}z. \end{aligned} \tag{21}$$

Here x is a dimensionless longitudinal coordinate; the $\chi(\varepsilon x)$ coefficient describes deviations of the wavenumber from the resonance value k_0 and may depend on the coordinate as a result of nonuniform density of the plasma. Finally, $\rho(\varepsilon x)$ is a beam–plasma coupling factor. That may change together with the transverse geometric structure of the system [12].

Equation set (21) reduces to the third-order equation

$$y''' - \mathrm{i}\chi(\varepsilon x)y'' + \frac{\mathrm{i}}{2}\rho(\varepsilon x)y = 0. \tag{22}$$

Its characteristic equation is

$$l = p^3 - \mathrm{i}\chi p^2 + \frac{\mathrm{i}}{2}\rho = 0. \tag{23}$$

Equation (23) is characterized by a singularity if $\chi = \chi_c = -\frac{3}{2}\rho^{\frac{1}{3}}$. That is the point where two roots merge, which corresponds to a simple turning point for (22). At $\chi > \chi_c$, one of the roots of (23) has a positive real part, such that the corresponding solution grows exponentially due to the PBI. With $\chi < \chi_c$ all the three roots are purely imaginary, and the corresponding waves are stable. It is this domain of parameters which is of special interest for the effect under consideration.

Figure 3 shows results of numerical solution of equation (22) with $\rho = \rho_0 + e_1 \cos(\varepsilon x)$ and $\chi = \chi_0 + e_2 \sin(\varepsilon x)$, $\chi < \chi_c$, and the corresponding spectra. It is seen that for the motion of the point (ρ, χ) around a circle in a specified direction one of the waves grows, while the other decays and the third remains nearly unchanged. If the direction of motion is altered, then the growing wave decays and the damping one grows.

It should be noted that the example given in this section is of illustrative character and of poor fit for implementation. We have considered a strictly monochromatic signal of a fixed frequency ω which determines the necessary variation domain for the parameter $\chi < \chi_c$. In actual fact, the spectrum always contains small 'background' components that may correspond to $\chi > \chi_c$, i.e. to the PBI. Besides, the variations of parameters χ and ρ ought to be realized over a long interval (much longer than ε^{-1}) of the x-axis. It might prove impossible to effectuate such variations in reality.

Nevertheless, the example is important because it demonstrates, in principle, the possibility of a geometric instability in physical systems. In addition, it will help to define classes of systems in which the effect of geometric instability is possible or impossible. We shall discuss it further.

5. Discussion

We have considered the effect of geometric instability which may arise in dynamic systems as a result of independent variation of several parameters. The effect consists of an exponential growth of solutions of linear differential equations with slowly variable coefficients. The eigenvalues of the equation may correspond to the steady-state, i.e. restricted solutions. The exponential growth (or attenuation) of solutions arise due to an anholonomic geometric component in the exponent of the WKB solutions of the homogeneous linear ordinary differential equations. Note that parameter variations are considered far from singularities of the initial equation and in the absence of parametric resonances. This justifies the application of the WKB approximation. Let us list the main properties of the effect discovered.

(a) *Geometricity*. The change of the solution amplitude accompanying adiabatic variation of the parameters does not depend on the rate of their variation, being determined solely by the trajectory of the representative point in the parameter space. Because of this property

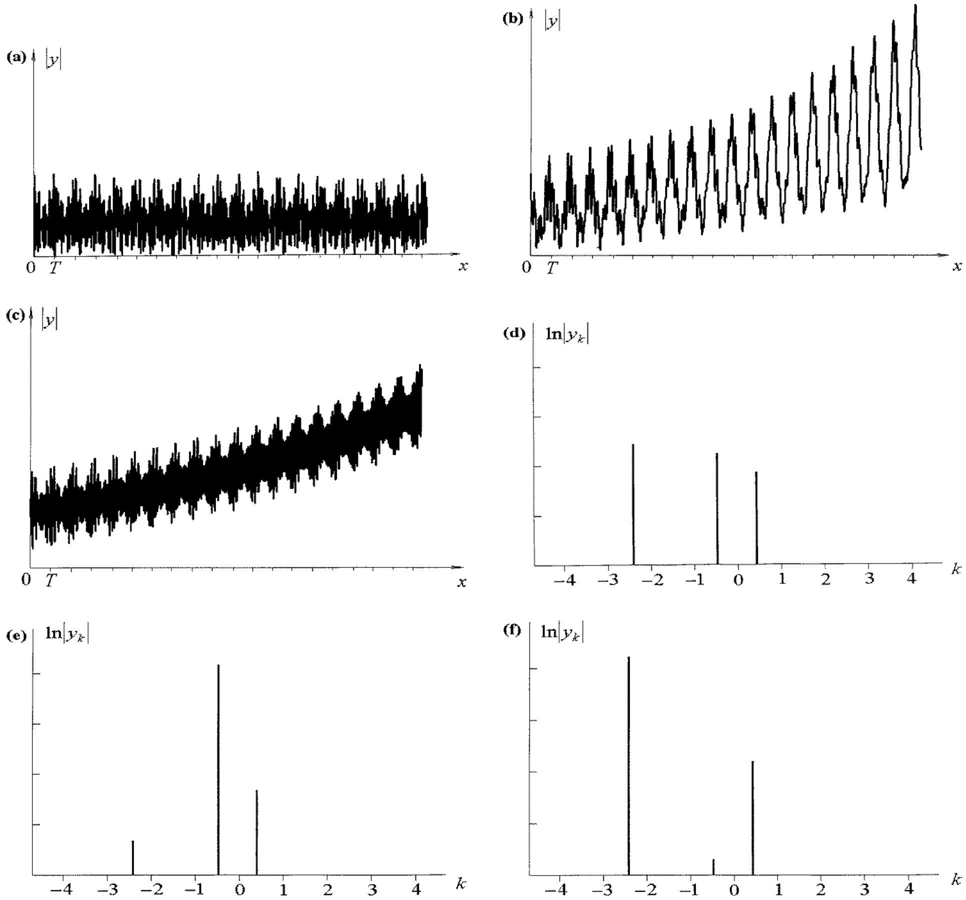

Figure 3. (*a*)–(*c*) The amplitudes $|y(x)|$ of solutions to equation (22) for different variable parameters $\rho = 1 + e_1 \cos(\varepsilon x)$, $\chi = -2.5 + e_2 \sin(\varepsilon x)$. The small-scale oscillations correspond to a sum of three complex WKB solutions with different amplitudes and progressive phases $\int^x p_j(\varepsilon\xi)\,\mathrm{d}\xi$ (p_j are the imaginary roots of the characteristic equation (23)). The period of large-scale oscillations coincides with that of parameter variations, $T = 2\pi\varepsilon^{-1}$. (*d*)–(*f*): Spatial spectra $y_k(k) = \frac{1}{2\pi}\int y(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x$ of solutions to equation (22) for fixed coefficients $\chi = -2.5$, $\rho = 1$, upon alteration of coefficients at a certain x-interval in accordance with (*a*)–(*c*), respectively. The wavenumbers $k_j = -\mathrm{i}p_j$ correspond to different oscillation modes, specifically: $k_{1,2} \approx \pm 0.5$—two beam waves and $k_3 \approx -2.5$—the plasma wave. (*a*), (*d*) $e_1 = 0.5$, $e_2 = 0$, $\varepsilon = 0.2$. The point (ρ, χ) moves along a straight line and does not cover any area in the parameter space. Accordingly, the phase Φ_{j0} gained over one period is equal to zero and the amplitudes of the three independent WKB solutions do not change. (*b*), (*e*) $e_1 = 0.5$, $e_2 = -0.5$, $\varepsilon = 0.2$. The point (ρ, χ) moves clockwise around a circle. This results in an exponential growth of one beam wave and exponential decay of the plasma wave. (*c*), (*f*) $e_1 = 0.5$, $e_2 = 0.5$, $\varepsilon = 0.2$. The point (ρ, χ) moves counterclockwise around the same circle, bringing forth an exponential growth of the plasma wave and exponential decay of one beam wave.

the geometric instability is less sensitive to the time dependence of the parameters then, say, the parametric resonance instability. In the latter case small deviations from the resonance dependence are capable of breaking the instability.

(b) *Anholonomicity*. When a representative point goes around a closed contour in the parameter space, the WKB gains a nonzero complex phase. This leads to a change of the solution amplitude, implying some work done over the system which does not return to its initial state. Because of the geometricity, a continuous motion of the representative point changes the solution amplitude by the same factor over each revolution. Hence, the solution will grow or damp exponentially.

(c) *Reciprocity of the instability.* This property follows from the anholonomicity and geometricity of the effect. Let the solution increase exponentially when the representative point moves around a certain contour in parameter space. This growth may be turned into attenuation by changing the parameters in such a way that the representative point should go around the same contour in the opposite direction (see section 3.1).

(d) *Smallness of the effect.* It was shown in section 2, that the growth rate of the geometric instability is of the same order as ε (small adiabatic parameter). Moreover, if the amplitude of parameter variations is small, then the geometric instability growth rate is proportional to the square of the amplitude.

The above considered mathematical examples (section 3.1 and 3.2) show that the geometric instability may manifest itself essentially, even in the simplest of differential equations. This is possible assuming that the instability is pertinent to a broad class of equations. Nevertheless, it was not a simple matter to find an example of a physical system in which the effect of geometric instability would be observable. The simplest systems which may be represented as a set of coupled oscillators do not exhibit the effect (see section 4.1), whereas the similar effect of geometric Berry–Hannay phase is observed even in a single generalized oscillator, or one with attenuation [3, 9]. The example of the plasma–beam system (section 4.2) confirms in principle, the possibility of geometric instability in physical systems. So, what is the reason for its absence in simple systems, such as coupled oscillators? And what is special about the plasma–beam system which is equally representable as a set of coupled moving oscillators?

The following explanation seems reasonable. In Hamiltonian mechanical systems describable in terms of a Hermitian Hamilton operator or a real Hamiltonian, it is always possible to construct an adiabatic invariant of action [5, 13]. The presence of an adiabatic invariant prevents the appearance of a geometric instability because it limits the solution amplitude through limited variations of the parameters. As is known, a set of damping oscillators is a Hamiltonian system (if [9]), and hence the absence of a geometric instability in the system confirms the validity of our arguments. The plasma–beam system is equally a Hamiltonian system (it is possible to construct an electrodynamic Hamiltonian for the particles and waves). But the analogy with coupled moving oscillators is not complete. Any mechanical system is described by a Hermitian Hamilton operator, while the Hamiltonian of the plasma–beam system is non-Hermitian. Really, the eigenvalues (frequencies or wavenumbers) of beam oscillations are all shifted by the same imaginary value due to the Doppler effect. Thus, it is impossible to define a mechanical real Hamiltonian for the system and generally speaking, it is impossible to construct an adiabatic invariant. This seems to explain the existence of a geometric instability in the system.

It follows from these arguments that a geometric instability may be observed in the physical systems which do not possess an adiabatic invariant, i.e. in non-Hamiltonian systems or in nonmechanical Hamiltonian systems.

This paper shows, above all, the existence of a novel effect of geometric instability which may arise in different dynamical systems. Also described are the main properties of the effect, some of which only qualitatively. In particular, the problem of real physical systems in which the geometric instability can be observed, remains. In addition it would be interesting to give a more rigorous classification of dynamical systems from the geometric instability point of view, i.e. to formulate the necessary and sufficient conditions for the existence of geometric instability. It would be important to extend the theory to nonlinear, multi-dimensional systems and to non-adiabatic parameter variations, as was done for the Berry–Hannay geometric phases.

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Appendix. The Hannay geometric phase in the generalized oscillator

In order to illustrate application of the general WKB formalism, let us analyse the example of a generalized oscillator in which the Hannay geometric phase arises. The Hamiltonian of such an oscillator is of the following form:

$$H = \tfrac{1}{2}(x(\varepsilon t)Q^2 + 2y(\varepsilon t)QP + z(\varepsilon t)P^2)$$

where Q and P are the generalized coordinate and momentum, respectively, and x, y, z are adiabatically varying parameters with $xz > y^2$. Let us write the equation of motion corresponding to this Hamiltonian,

$$Q'' - \frac{z'}{z}Q' + \left(xz - y^2 + y\frac{z'}{z} - y'\right)Q = 0.$$

The derivatives are taken here with respect to t. By comparing this equation with equation (1) to (3) it is easy to obtain the parameters and magnitudes used in the general formalism

$$\begin{array}{lll} n = 2 & s = 3 & q = (x, y, z) \\ \tilde{a}_{10} = 0 & \tilde{a}_{20} = xz - y^2 & \\ \tilde{b}_{11} = 0 & \tilde{b}_{12} = 0 & \tilde{b}_{13} = -\frac{1}{z} \\ \tilde{b}_{21} = 0 & \tilde{b}_{22} = -1 & \tilde{b}_{23} = \frac{y}{z}. \end{array}$$

The characteristic equation with its approximate roots is (see (4) and (6)):

$$l = l_0 + l_1 = p^2 + (xz - y^2) - \frac{z'}{z}p + \left(y\frac{z'}{z} - y'\right) = 0$$

$$\tilde{p}_{j0} = \pm \mathrm{i}\sqrt{xz - y^2} \equiv \pm \mathrm{i}\omega.$$

Now, by substituting these expressions into (11) we can derive the geometric phase Φ_j of the solution and single out its integrable (holonomic) and nonintegrable (anholonomic) components, namely:

$$\begin{aligned} \Phi_j &= \int \left[\left(-\frac{z}{4\omega^2}\right)\mathrm{d}x + \left(\frac{y}{2\omega^2} \mp \mathrm{i}\frac{1}{2\omega}\right)\mathrm{d}y + \left(-\frac{x}{4\omega^2} + \frac{1}{2z} \pm \mathrm{i}\frac{y}{2\omega z}\right)\mathrm{d}z\right] \\ &= \int \frac{-z\,\mathrm{d}x + 2y\,\mathrm{d}y - x\,\mathrm{d}z}{4(xz - y^2)} + \int \frac{\mathrm{d}z}{2z} \pm \mathrm{i}\int \frac{y}{2\omega}\left(\frac{\mathrm{d}z}{z} - \frac{\mathrm{d}y}{y}\right) \\ &= \ln\left(\sqrt{\frac{z}{\omega}}\right) \pm \mathrm{i}\int^t \frac{y}{2\omega}\left(\frac{z'}{z} - \frac{y'}{y}\right)\mathrm{d}\tau. \end{aligned}$$

This will yield, according to (10), two independent WKB solutions for the generalized oscillator, i.e.

$$Q_j = \sqrt{\frac{z}{\omega}}\exp\left\{\pm \mathrm{i}\int^t \omega\,\mathrm{d}\tau \pm \mathrm{i}\int^t \frac{y}{2\omega}\left(\frac{z'}{z} - \frac{y'}{y}\right)\mathrm{d}\tau\right\}.$$

The first term in the exponent is the usual dynamic phase, while the second is Hannay's geometric phase. It is purely imaginary and hence does not change the solution amplitude. The amplitude is connected to parameter variations by the adiabatic invariant which can be easily constructed using the pre-exponential factor in the WKB solutions,

$$|Q(t)|^2 \frac{\omega(t)}{z(t)} = \text{Inv}.$$

The expressions derived for the dynamic phase, the Hannay geometric phase and the adiabatic invariant coincide completely with the results which were obtained earlier by other methods [3, 9].

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